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On Polynomial δ-Type Functions and Approximation by Monotonic Polynomials

ZALMAN RUBINSTEIN*

Clark University, Department of Mathematics, Worcester, Massachusetts 01610

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Some attention has been given in recent years to the problem of approximation of real functions by monotone polynomials. In 1965, O. Shisha [5] proved, among other things, the following result: If $1 \leqslant k \leqslant p$ and if a real function $f(x)$, defined on $[0, 1]$, satisfies

$$f^{(k)}(x) \geqslant 0, \qquad |f^{(p)}(x)| \leqslant M, \qquad \text{for} \quad 0 \leqslant x \leqslant 1,$$

then for every integer $n(\geqslant p)$, there exists a real polynomial $Q_n(x)$ of degree not exceeding n such that the inequalities

$$Q_n^{(k)}(x) \geqslant 0$$

and

$$|f(x) - Q_n(x)| \leqslant \frac{C}{n^{p-k}} w\left(f^{(p)}, \frac{1}{n}\right) \tag{1}$$

hold for all $0 \leqslant x \leqslant 1$, where C depends only upon p and k and $w(\phi, h)$ is the modulus of continuity of the function ϕ.

Roulier [4], and Lorentz and Zeller [2] continued the investigation in this direction by relaxing, somewhat, the conditions on f, and by sharpening the estimate (1), particularly for large n.

In this note, we deal briefly with a related problem, namely, that of uniform approximation of a monotone continuous real-valued function by monotone polynomials, which, in addition, agree with the function on a finite set of points. Subsequently, we mention some applications.

The main theorems of this note are:

THEOREM A. *Let* $0 < x_1 < \cdots < x_n \leqslant 1$ *and* $0 < y_1 < \cdots < y_n$ *be fixed. There exists a polynomial* $Q(x)$ *such that*

(a) $Q(0) = 0$, $Q(x_i) = y_i$, $i = 1, 2,..., n$;

(b) $Q'(x) \geqslant 0$ *for all real* x.

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THEOREM B. *Any strictly increasing continuous function defined on* [0, 1], *can be uniformly approximated, as closely as desired, by a strictly increasing polynomial and in such a way that the two functions agree on an arbitrarily given finite set of points.*

In the proof of Theorem A, we shall need the following:

LEMMA 1. *Let* r, $0 < r < 1$, *be a rational number. The polynomial*

$$P_r(x) = \frac{x^k(1-x)^{m-k}}{r^k(1-r)^{m-k}},$$

where $r = k/m$ (k *and* m *positive even integers*), *has the following properties*:

(a) $P_r(0) = P_r(1) = 0$, $P_r(r) = 1$, $P_r(x) \geqslant 0$ *for all real* x.

(b) $P_r(x)$ *is strictly increasing in the interval* $[0, r]$ *and strictly decreasing in the interval* $[r, 1]$.

LEMMA 2. *The polynomials*

$$Q_{r,n}(x) = c_n P_r^n(x), \qquad n = 1, 2, \ldots,$$

where

$$c_n = \left(\int_0^1 P_r^n(x)\,dx\right)^{-1} = (mn+1)\binom{mn}{kn} r^{kn}(1-r)^{mn-kn},$$

have the following properties:

(a) $Q_{r,n}(0) = Q_{r,n}(1) = 0$, $Q_{r,n}(r) = c_n$, $Q_{r,n}(x) \geqslant 0$ *for all real* x.

(b) $Q_{r,n}(x)$ *is strictly increasing in the interval* $[0, r]$ *and strictly decreasing in the interval* $[r, 1]$.

(c) $\int_0^1 Q_{r,n}(x)\,dx = 1$.

(d) $\lim_{n\to\infty} Q_{r,n}(x) = 0$ *for* $x \in [0, 1]$, $x \neq r$.

Proof. (a), (b), and (c) follow immediately by Lemma 1 and the definition of c_n. To verify (d), we notice that by the binomial formula,

$$\binom{mn}{kn} r^{kn}(1-r)^{mn-kn} < 1.$$

Therefore, $c_n < mn + 1$ and, since $0 \leqslant P_r(x) < 1$ for $x \in [0, 1]$, $x \neq r$, it follows that $\lim_{n\to\infty}[c_n P_r^n(x)] = 0$.

Remark. Actually, by performing a more rigorous calculation with the help of Stirling's formula, one obtains the asymptotic estimate

$$c_n \sim [2\pi r(1-r)]^{-1/2}(mn)^{1/2}.$$

Proof of Theorem A. Consider the polynomial

$$Q(x) = \sum_{j=1}^{n} \alpha_j \int_0^x Q_{r_j,n_j}(t)\, dt, \tag{2}$$

where α_j, r_j and n_j are defined below and where $Q_{r_j,n_j}(x)$ is as in Lemma 2.

It is clear that $Q(0) = 0$ and that $Q'(x) \geqslant 0$ for all real x if all α_j are $\geqslant 0$.

Set $y = (y_1, y_2, \ldots, y_n)$, $c_{ij} = \int_0^{x_i} Q_{r_j,n_j}(t)\, dt$, $u_j = (c_{1j}, c_{2j}, \ldots, c_{nj})$, $i, j = 1, 2, \ldots, n$.

To satisfy condition (a), we have to solve the system

$$\sum_{j=1}^{n} \alpha_j u_j = y. \tag{3}$$

The point $y \in R_n$ is a strictly increasing sequence of positive numbers and for each j, the point $u_j \in R_n$ is a strictly increasing sequence of positive numbers bounded above by unity. It follows that the point y lies interior to the infinite "wedge" in R_n with vertex at the origin, spanned by the vectors $(0, 0,\ldots, 0, 1)$, $(0, 0,\ldots, 0, 1, 1),\ldots, (1, 1,\ldots, 1)$, which we shall denote by $e_1, e_2, \ldots, e_n$, respectively. One deduces that the system (3) will have a strictly positive solution $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ (that is, one with all $\alpha_j > 0$) provided the vectors u_j are sufficiently close to the vectors e_j. To this end, we choose the δ-type polynomials $Q_{r_j,n_j}(x)$ in the following manner. For a given j, $1 \leqslant j \leqslant n$, and a given positive $\epsilon < 1$, we select an r_j such that $x_{n-j} < r_j < x_{n-j+1}$ and construct a polynomial $Q_{r_j,n_j}(x)$ such that

$$Q_{r_j,n_j}(x) < \epsilon \quad \text{for} \quad 0 \leqslant x \leqslant x_{n-j} \quad \text{and for} \quad x_{n-j+1} \leqslant x \leqslant x_n .$$

It follows that

$$0 < c_{ij} < \epsilon \quad \text{for} \quad 1 \leqslant i \leqslant n-j;$$

$$1 - \epsilon < c_{ij} \leqslant 1 \quad \text{for} \quad n-j+1 \leqslant i \leqslant n.$$

It is clear that $u_j \to e_j$ as $\epsilon \to 0$.

Now, the linear system

$$\sum_{j=1}^{n} \beta_j e_j = y$$

has the positive solution

$$(\beta_1, \beta_2, \ldots, \beta_n) = (y_n - y_{n-1}, y_{n-1} - y_{n-2}, \ldots, y_2 - y_1, y_1).$$

By continuity, it follows that for sufficiently small $\epsilon = \epsilon(y)$, the linear system (3) has a positive solution $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. This concludes the proof.

Theorem B is, of course, a direct consequence of Theorem A.

We shall mention, now, a few consequences of the previous results.

COROLLARY 1. *Let* $J = [x_0, y_0]$ $(0 \leqslant x_0 < y_0 \leqslant 1,\ x_0 + y_0 < 1)$. *For every positive number M there exists a polynomial* $P(x)(\not\equiv 0)$ *which is non-negative for all x such that*

$$\int_J P(x)\,dx = M \int_{[0,1]-J} P(x)\,dx.$$

Proof. Let $P(x) = Q'(x)$, where $Q(x)$ is a polynomial as in Theorem A, satisfying

$$Q(0) = 0, \qquad Q(x_0) = \frac{\delta}{M+1}, \qquad Q(y_0) = \frac{M+\delta}{M+1}, \qquad Q(1) = 1,$$

with $0 < \delta < 1$ if $x_0 > 0$, $y_0 < 1$; $\delta = 0$ if $x_0 = 0$ and $\delta = 1$ if $y_0 = 1$. We deduce that

$$\int_J P(x)\,dx = Q(y_0) - Q(x_0) = \frac{M}{M+1}$$

and

$$M \int_{[0,1]-J} P(x)\,dx = M[Q(x_0) - Q(0) + Q(1) - Q(y_0)] = \frac{M}{M+1}.$$

Corollary 1 provides an elementary proof of the well-known

COROLLARY 2. *Let* $f(x)$ *be a bounded summable function on the interval* $[0, 1]$, *such that*

$$\int_0^1 x^k f(x)\,dx = 0, \qquad \text{for} \quad k = 0, 1, 2, \ldots .$$

Then $f(x)$ *vanishes at every point of continuity.*

Proof. Assume that $|f(x)| < M$ for $x \in [0, 1]$. Let x_1, $0 \leqslant x_1 \leqslant 1$, be a point of continuity of $f(x)$. If $f(x_1) \neq 0$ then we may assume that $f(x) > \eta > 0$ in some interval J as in Corollary 1, containing x_1.

By Corollary 1, there exists a polynomial $P(x)(\not\equiv 0)$ such that

$$\int_J P(x)\,dx = \frac{M}{\eta} \int_{[0,1]-J} P(x)\,dx.$$

In addition, we have

$$\int_J f(x)\,P(x)\,dx > \eta \int_J P(x)\,dx = M \int_{[0,1]-J} P(x)\,dx$$

and

$$\left| \int_{[0,1]-J} f(x)\,P(x)\,dx \right| < M \int_{[0,1]-J} P(x)\,dx.$$

We obtain a contradiction, since $\int_0^1 f(x)\,P(x)\,dx = 0$ implies that

$$\left| \int_{[0,1]-J} f(x)\,P(x)\,dx \right| = \int_J f(x)\,P(x)\,dx.$$

This completes the proof. In particular, if $f(x)$ is Riemann integrable, then it vanishes almost everywhere in the interval [0, 1].

It is interesting to relate Theorem A to a result which follows from a theorem due to Yamabe. He proved [6], using the Weierstrass approximation theorem, the following:

Given a continuous function $g(x)$ on [0, 1], n linear functionals Φ_i ($i = 1, 2,..., n$) on the space $C[0, 1]$ and a positive number ϵ, there exists a polynomial $p(x)$ such that

$$\max_{0 \leqslant x \leqslant 1} | g(x) - p(x)| < \epsilon$$

and

$$\Phi_i[g(x)] = \Phi_i[p(x)].$$

If this result is used, Theorem A (with (b) asserted only on [0, 1]) can be proved by choosing a function $g(x)$ such that

$$g(x) \geqslant c > 0 \qquad \text{for} \quad 0 \leqslant x \leqslant 1,$$

$$\int_0^{x_1} g(t)\,dt = y_1\,, \qquad \int_{x_{k-1}}^{x_k} g(t)\,dt = y_k - y_{k-1}\,, \qquad k = 2,..., n,$$

and defining the Φ_i by $\Phi_i(h) = \int_0^{x_i} h(t)\,dt$. The theorem then follows by Yamabe's result, with $\epsilon = c/2$.[1]

Normally, in a uniform approximation process by polynomials, it is desirable to carry over as many properties of the approximated function $f(x)$ as possible.

For example, the Bernstein polynomials

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1 - x)^{n-k} f\left(\frac{k}{n}\right)$$

[1] This proof and the reference to Yamabe's paper were suggested by the referee, to whom I am greatly indebted.

are strictly increasing on the interval [0, 1], if $f(x)$ is also. A similar result holds [3] if $f(x)$ is starlike in [0, 1] (that is, $f(\alpha x) \leqslant \alpha f(x)$ for $\alpha \in [0, 1]$, $x \in [0, 1]$). It would be, therefore, of interest to study properties which are inherited from a monotonic function to monotonic polynomials approximating it.

In the case of a three-point interpolation, one can actually construct a polynomial of the type of $Q(x)$ of Theorem A. Indeed, if $0 < a < b$, $0 < \rho < 1$, one verifies easily that the polynomial

$$R(x) = c[1 - x^n - (1 - x)^n] + bx^n,$$

where

$$c = \frac{a - b\rho^n}{1 - \rho^n - (1 - \rho)^n},$$

is strictly increasing and satisfies $P(0) = 0$, $P(\rho) = a$, and $P(1) = b$, provided n is a sufficiently large odd positive integer. It is enough to satisfy the conditions

$$1 - \left(1 - \frac{a}{b}\right)^{1/n} < \rho < \left(\frac{a}{b}\right)^{1/n},$$

since this implies that $0 < c < b$. Thus, the minimal admissible n depends only on ρ and the ratio a/b. It would be interesting to estimate the minimal degree of a $Q(x)$ satisfying (a) and (b) of Theorem A.

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